LECTURE SUMMARY 12.1

WEDNESDAY, JULY 27, 2016

Trajectory of linear system of differential equations

1. Definition of trajectory, equilibrium points.

2. **Theorem 1** The origin is an equilibrium point of the linear system of differential equations $\mathbf{z}' = \mathbf{A}\mathbf{z}$. Furthermore, if $\mathbf{A}$ only has nonzero eigenvalues, then the origin is the only equilibrium point.

Stability of the Origin

Theorem 2 Suppose that the eigenvalues of r_1 and r_2 of $\mathbf{A}$ are nonzero and have associated eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$, respectively. Then the equilibrium point at the origin is unstable if at least one eigenvalue is positive or is complex with positive real part. On the other hand, the origin is asymptotically stable if both eigenvalues are negative or are complex with negative real part.

The classification of the origin and the trajectories of the solutions may be described as follows:

Eigenvalues	Stability	Origin	Trajectories
$\lambda_1 > \lambda_2 > 0$	Unstable	Node	Along $\mathbf{v_2}$ near the origin and nearly parallel to $\mathbf{v_1}$ far from the origin
$\lambda_1 > 0 > \lambda_2$	Unstable	Saddle point	Hyperbolic in shape, pointing toward the line corresponding to $\mathbf{v_1}$
$0 > \lambda_1 > \lambda_2$	Asymptotically stable	Node	Along $\mathbf{v_1}$ near the origin and nearly parallel to $\mathbf{v_2}$ far from the origin
$\lambda_1 = \lambda_2 > 0$, $\mathbf{A}=\lambda_1\mathbf{I}$	Unstable	Proper Node	Lines pointing away from the origin
$\lambda_1 = \lambda_2 > 0$, $\mathbf{A} \neq \lambda_1\mathbf{I}$	Unstable	Improper Node	Difficult to draw by hand
$\lambda_1 = \lambda_2 < 0$, $\mathbf{A}=\lambda_1\mathbf{I}$	Asymptotically stable	Proper Node	Lines pointing towards the origin
$\lambda_1 = \lambda_2 < 0$, $\mathbf{A} \neq \lambda_1\mathbf{I}$	Asymptotically stable	Improper Node	Difficult to draw by hand
$\lambda_1, \lambda_2 = \alpha \pm \beta i$, $\alpha > 0$	Unstable	Spiral point	Outward spirals, either clockwise or counter-clockwise
$\lambda_1, \lambda_2 = \alpha \pm \beta i$, $\alpha < 0$	Asymptotically stable	Spiral point	Inward spirals, either clockwise or counterclockwise
$\lambda_1, \lambda_2 = \pm\beta i$	Stable	Center	Concentric ellipses or circles, either clockwise or counterclockwise